PERIOD–V

M12CH17

TOPIC

17

Transformations and Coordinates

17.1. TRANSFORMATIONS

When a plane figure undergoes a **transformation** it means its position or shape or size has changed.

The following types of transformation shall be considered in this chapter:

1. Reflection
2. Rotation
3. Translation
4. Enlargement
5. Special mappings

Note: Reflection, rotation and translation are called ***rigid motions*** because under such transformations the shape or size of the figure transformed does not change.

Enlargement is not a rigid because the size of the figure transformed is changed,

Note that the transformation of plane figures can be done in the Cartesian plane by transforming the *vertices* of the plane figure.

17.2. REFLECTION

A reflection is the image you see when you look in a mirror. The mirror forms the line of symmetry between between the object and the image.

Reflection conserves angles, lengths and area but reserves the figure. To define a reflection you need to know the position of the line which the figure is to be reflected.

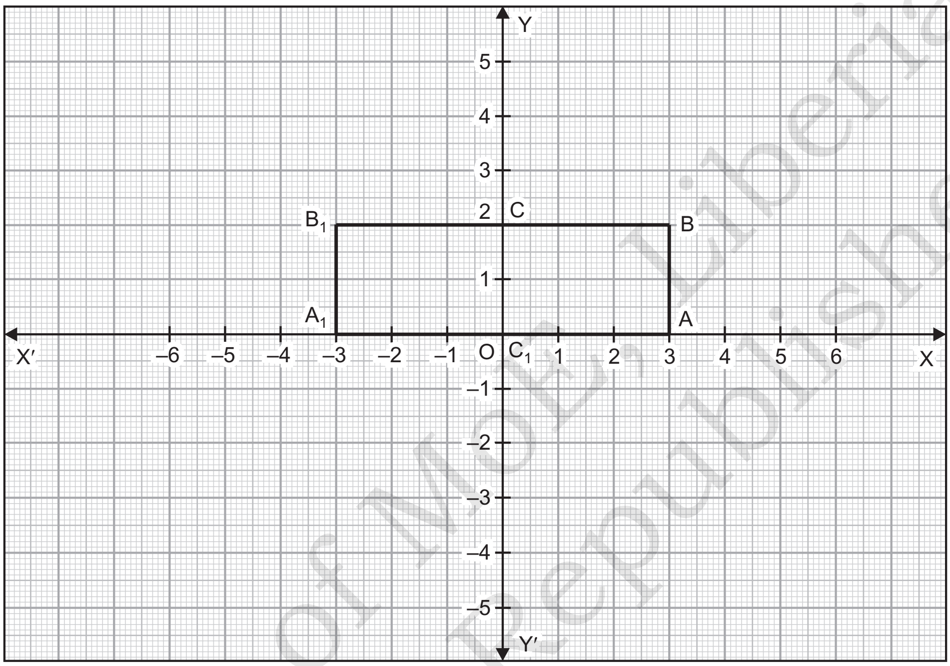
When a point is reflected in a line, the *image* point is at the opposite side of the line and the perpendicular distance from the point to the line is equal to the perpendicular distance from the image point to the line. The line called the *mirror line* or *line of reflection*.

i.e. object distance from mirror line = image distance from mirror line.

In figure OABC has been reflected in the y-axis to give O, A, B, C.

We shall consider reflections in the following mirror lines.

1. The x-axis or the line $y = 0$.
2. The y-axis or the line $x = 0$.



1. Reflection in the x-axis (i.e. reflection in the line $y = 0$):

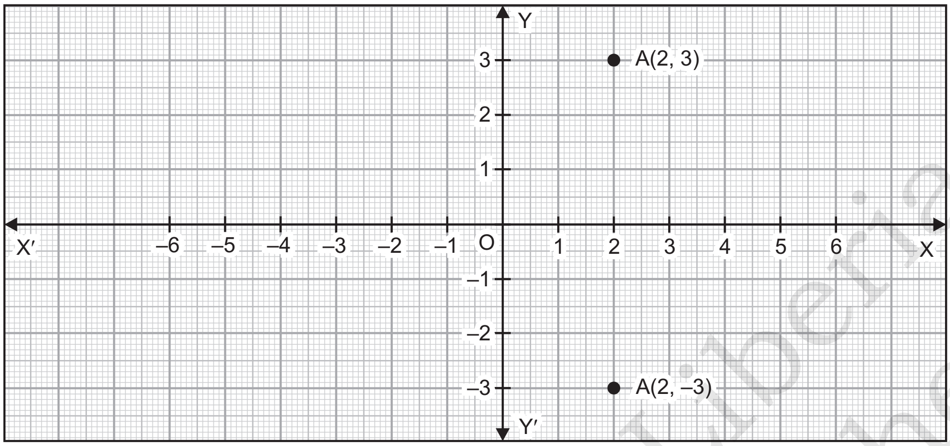
For reflection in the x-axis (horizontal axis), the x-axis serves as the mirror line.

Example 1. *Reflect the point X(2, 3) in the x-axis.*

Solution. Steps

1. With your x and y axes drawn. Locate the point A(2, 3) on the graph.
2. Measure the perpendicular distance from the mirror line (i.e., x-axis) to the point.
3. Measure the same distance as at the opposite side of the x-axis and locate your image point.
4. Write the coordinates to get your image A(2, –3) i.e., the x-coordinate is 2 and the y-coordinate is – 3. See Fig. for the graph.

Note: Write the x-coordinate first, followed by the y-coordinate.



Example 2. *A is the image of A(–2, –5) under a reflection in the x-axis. Find the coordinates of A.*

Solution. Under reflection in the *x*-axis, $(x, y) \rightarrow (x, -y)$

i.e. A(–2, –5) → A(–2, 5)

The coordinates of A is (–2, 5)

2. Reflection in the *y*-axis (*i.e.*, reflection in the line *x* = 0):

For reflection in the *y*-axis, the *y*-axis (vertical axis) serves as the mirror line.

Example 3. *Reflect the point C(2, 3) in the y-axis.*

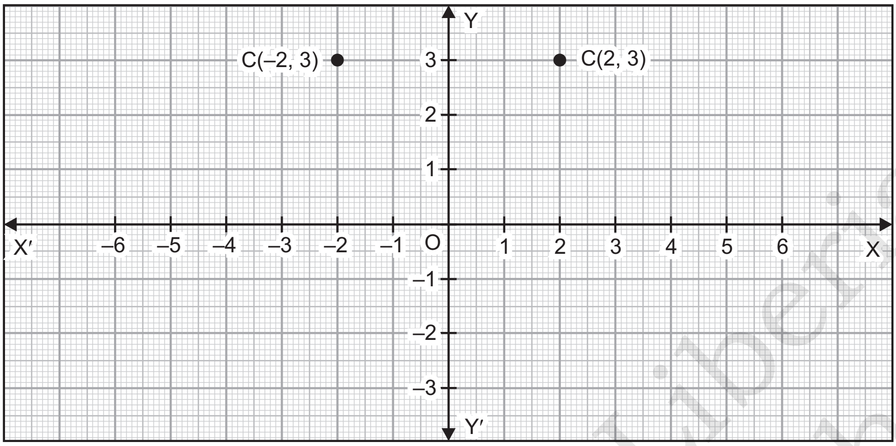
Solution. First method

For example when C(2, 3) is reflected in the *y*-axis we have C(– 2, 3) as the image point.

The steps are the same as reflected in the *x*-axis but here the *y*-axis serves as the mirror line.

Steps

1. With your axis drawn, locate the point C(2, 3) on the graph.
2. Measure the perpendicular distance from the mirror line (i.e., *y*-axis) to the point.
3. Measure the same distance at the opposite side of the *y*-axis and locate the image.
4. Write the coordinates to get your image C(– 2, 3). See figure for for the graph.



Example 4. *Using a scale of 2 cm to 1 unit, draw x and y axes for the interval* $0 \leq x \leq 8$ *and* $-6 \leq y \leq 6$.

(*i*) *Plot the points A(3, 1), B(1, 1) and C(1, 5) and described ABC.*

(*ii*) *Draw the* $\Delta A_1B_1C_1$ *which is the image of* ΔABC *under the reflection in the x-axis where* $A \to A_1$, $B \to B_1$, $C \to C_1$. *Indicate clearly the coordinates of* $\Delta A_1B_1C_1$.

Solution. Using the scale, draw the *x* and *y* axes in the intervals given on a graph sheet.

(*i*) Plot the points A, B and C on the graph and indicate clearly their coordinates. Join these points with straight lines to get ΔABC as shown in figure. ABC is a right-angled triangle.

(*ii*) The mapping for reflection in the *x*-axis is:

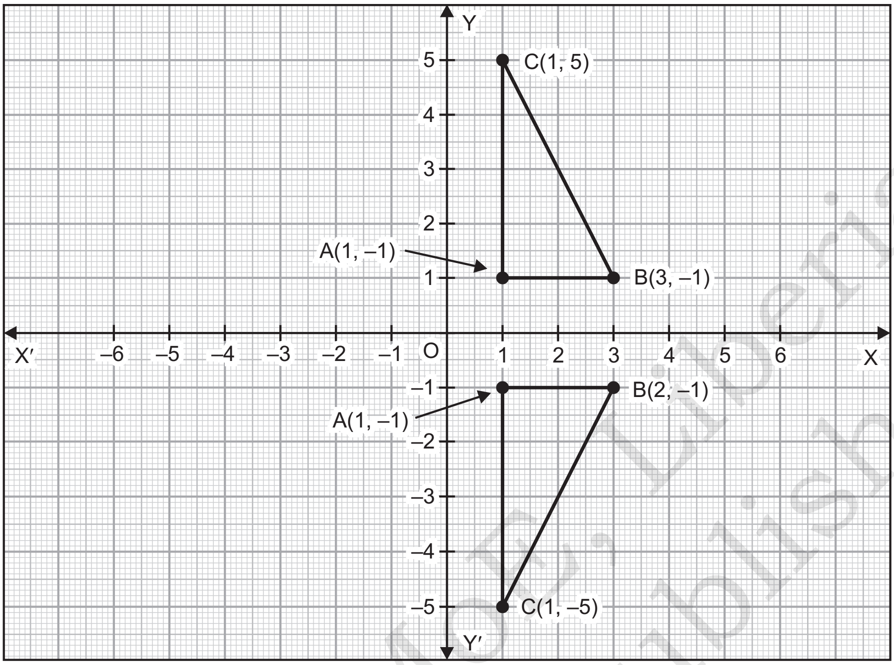
$$\begin{pmatrix} x \\ y \end{pmatrix} \text{ a } \begin{pmatrix} x \\ -y \end{pmatrix} \text{ or } (x, y) \to (x, -y)$$

$A(3, 1) \to A_1(3, -1)$

$B(1, 1) \to B_1(1, -1)$

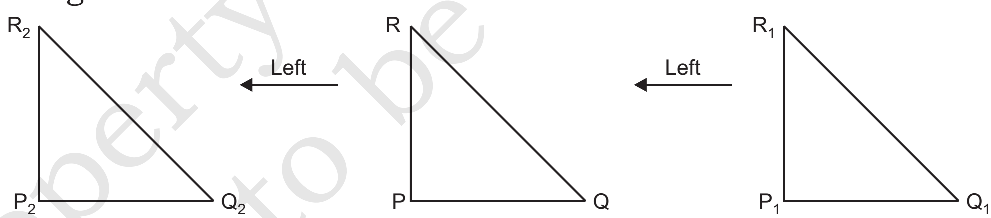
$C(1, 5) \to C_1(1, -5)$

Plot the points A_1, B_1 and C_1 on the graph to get $\Delta A_1B_1C_1$ and indicate their coordinates clearly as shown in figure.



Translation

If a figure is moved in a straight line by certain amounts in a certain direction without turning, and it still looks the same, the figure is said to have undergoen translation. Angles, lengths and areas remain unchanged.



The acitivity above shows that the movement of triangle ABC is a translation. When you walk from one place to another, you are making a translation. When you push a table away from a place or when you lift a bucket of water, you are making a translation. Give four real life examples of translation.

Translation of points

Translation can be described by a vector.

To translate a point P(2, 1) by a translation vector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, first plot the point on a graph sheet as shown in figure. Next move P 1 unit horizontally and 3 units vertically to the position P_1. The coordinates of this new position is (3, 4).

We can also use the relation below to find the coordinates of the new position. If the point (x, y) is translated by the vector $\begin{pmatrix} a \\ b \end{pmatrix}$.

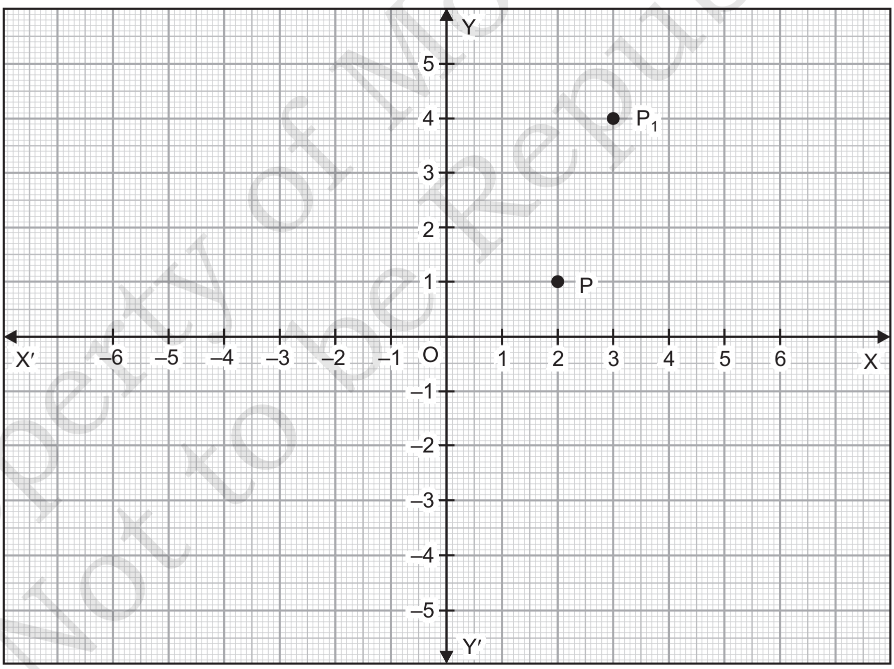
The mapping is:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x+a \\ y+b \end{pmatrix}$$

or $$(x, y) \to (x + a, y + b)$$

The vector $\begin{pmatrix} a \\ b \end{pmatrix}$ vector will be given.

The image point is therefore found by adding the vector to the position vector of the point.



1. Find the image when the point and the translation vector are given.

Example 5. *Find the image A′ if A(3, 4) is translated by the vector* $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$.

Solution. Under a translation by the vector $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$,

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} d \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 + (-2) \\ 4 + 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix} \text{ or } (1, 9)$$

∴ The image of (3, 4) under the translation is (1, 9).

2. Finding the point when its image and the translation vector are given.

Example 6. *P(4, 6) is the image of a point P under the translation by the vector* $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. *Find the point P.*

Solution. If P has coordinates (x, y) then under a translation by the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, then

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} x + 1 \\ y + 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

Therefore from equality of vectors

$$x + 1 = 4 \text{ and } y + 2 = 6 \; □ \; x = 3 \text{ and } y = 4$$

∴ The point P has coordinates (3, 4).

Translation of plane figures

The translation of plane figures can be done in the Cartesian plane by translating the vertices of the plane figure by the given translation vector.

Example 7. *(a) Using a scale of 2 cm to 1 unit on each axies draw on a graph sheet two perpendicular axes OX and OY.*

(*b*) *On this graph, mark the x-axis from – 5 to 5 and the y-axis from – 5 to 5.*

(*c*) *Plot the point A(– 1, – 1), B(3, 4) and c(2, 1). Join the points to form a triangle.*

(*d*) *Draw the image of the triangle ABC under the translation by the vector* $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$ *and* $A \rightarrow A_1$, $B \rightarrow B_1$ *and* $C \rightarrow C_1$.

Solution. (*a, b*) Draw to perpendicular axes todivide the graph sheet into two equal parts and label each axis as shown in figure.

(*c*) Locate the points A(–1, –1), B(3, 4) and C(2, –1) and join them to get triangle ABC.

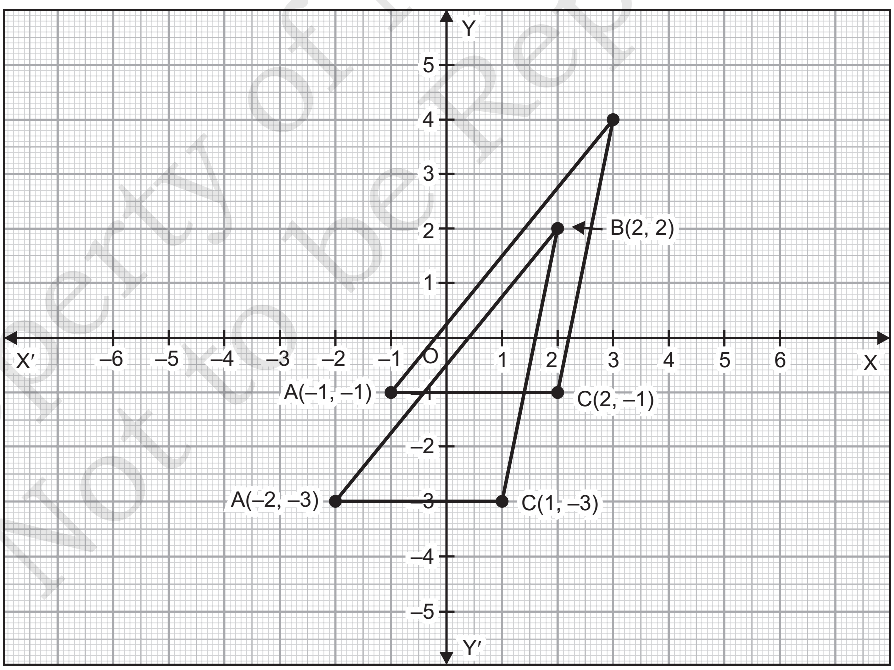
(*d*) Translate the points A, B and C by vector $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$ to get the image points A_1, B_1 and C_1.

$$A\begin{pmatrix} -1 \\ -1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \quad \text{i.e.} \quad A_1(-1, -3)$$

$$B\begin{pmatrix} 3 \\ 4 \end{pmatrix} \rightarrow \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{i.e.} \quad B_1(2, 2)$$

$$C\begin{pmatrix} 2 \\ -1 \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \quad \text{i.e.} \quad C_1(1, -3)$$

Locate the points A_1(–1, –3), B_1(2, 2) and C_1(1, –3) on the graph and join them to get triangle $A_1B_1C_1$.



Enlargement from the Origin O

If the point (*x, y*) is enlarged from the origin by a scale factor *k*, then image point is (*kx, by*)

The mapping is: $\begin{pmatrix} x \\ y \end{pmatrix} \to k\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ ky \end{pmatrix}$ or $(x, y) \to (kx, ky)$

k may be a positive or negative whole number or fraction.

Example 8. *Find the image of (3, 4) under the enlargement from the origin with scale factor –2.*

Solution. $\begin{pmatrix} 3 \\ 4 \end{pmatrix} \to 2\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -6 \\ -8 \end{pmatrix}$ or (–6, –8).

EXERCISE

1. A′ is the image of A(– 2, 4) under a reflection in the y-axis. Find the coordinates of A′.
2. Find the image of (*i*) (1, 2) and (*ii*) (–2, –4) under the translation by the vector $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$.
3. P(–2, 4) is the image of a point P(1, 2) under a translation by a vector. Find the translation vector.
4. The point (3, 4) is translated by the mapping $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 3x \\ y \end{pmatrix}$. Find its image.
5. Find the images of the following points under the mapping $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x + y \\ 3x + 2y \end{pmatrix}$

   (*i*) (1, 2) (*ii*) (–2, 3)
6. Find the image of (3, –7) under the trnasformation $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2x + y \\ y - 3x \end{pmatrix}$.